# CUBA-ESPRIT FOR ANGLE ESTIMATION WITH CIRCULAR UNIFORM BEAM ARRAYS

**Andreas Richter[(1)], Reiner S. Thomä[(2)]**

***Ilmenau University of Technology***
***Deparment of Communication and Measurement Technology***
***98694 Ilmenau, POB 100565***
***Germany***
[(1)]***Email: ric@e-technik.tu-ilmenau.de***
[(2)]***Email: tho@e-technik.tu-ilmenau.de***

## INTRODUCTION

Currently, spatial signal processing for adaptive source separation and interference cancellation are extensively discussed for application in future mobile radio systems. ESPRIT-type algorithms are especially concerned because of their low arithmetic complexity and their inherent ability for superresolution estimation of the direction of arrival (DoA) [1]. The latter is also true for mobile radio vector channel sounding [5] and radio surveillance. The effectivity and ability of the applied algorithms mainly depends on the antenna array geometry. Mostly, uniform linear arrays (ULAs) are discussed because of their regular structure (which simplifies ESPRIT application) and their ability to cope with coherent sources which are typical for any mobile radio multipath environment. But, if 360° azimuth DoA coverage is claimed, at least two cross-arranged ULAs are required. Unfortunately, this is not enough to handle coherent sources. Therefore, uniform rectangular or uniform circular arrays are generally considered. On the other hand, there exist very effective, circular uniform multibeam antenna designs (**c**ircular **u**niform **b**eam **a**rrays, CUBA). To be specific, a CUBA consists of $N$ antennas with circular symmetric beam patterns $b_0(\theta)$. The main beam directions of two adjacent elements are rotated by $\theta_0 = 2\pi/N$, respectively (Fig.1).

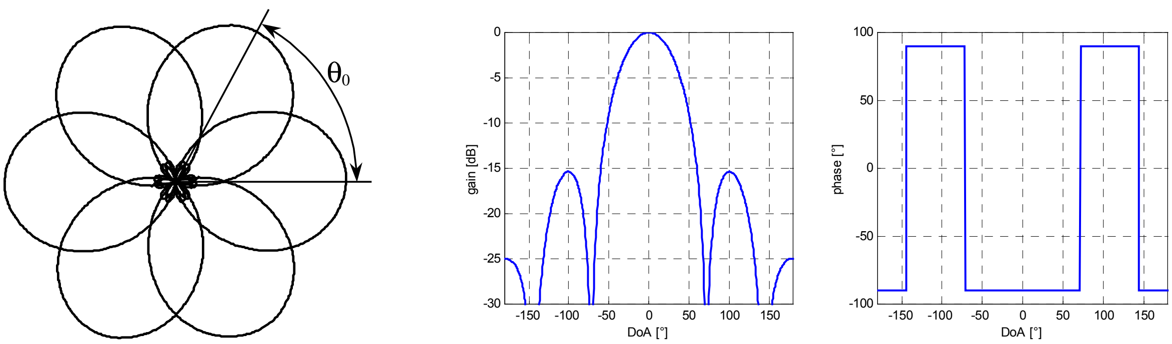

Fig. 1. Beam patterns of a 6 element CUBA (left), gain and phase of a single element (right).

An example CUBA antenna is described in [6]. The advantage of this multibeam antenna is a very compact mechanical design, since it allows elevation beam shaping with a single, bi-conical horn structure. However, the well known standard ESPRIT-algorithm cannot be applied for CUBA antennas. Therefore, the he aim of this paper is to describe a new ESPRIT-type algorithm for CUBAs which can even handle coherent sources. In section II the basics of the algorithm are given. Since in reality, all arrays suffer from systematic errors, in section III a procedure for the estimation of required transformation matrix is presented that serves also for calibration purposes. Simulation results of the developed CUBA-ESPRIT algorithm are described in section IV.

## CUBA-ESPRIT

Given an narrow band signal from a single source that is impinging on the CUBA with the angle $\theta_p$, the output signals of the $N$ array elements can be described by means of the sampled beam pattern as

$$b_p(n\cdot\theta_0)=\gamma_p\cdot b_0(n\cdot\theta_0-\theta_p) \tag{1}$$

with $\theta_0=2\pi/N$ and $n=0\ldots(N-1)$. Since the antenna beam pattern is periodic in 2π

$$b_0(\theta)=b_0(\theta+k\cdot 2\pi),\ k\in Z\,, \tag{2}$$

the equivalent virtual aperture function of a single element is defined by the discrete Fourier transform:

$$g_0(l\cdot s_0)=\sum_{n=0}^{N-1}b_0(n\cdot\theta_0)\mathrm{e}^{-\mathrm{j}2\pi\frac{nl}{N}}\ \text{ with }\ s_0=1/(2\pi)\,. \tag{3}$$

Consequently, in the virtual aperture domain the single source response (1) follows as

$$g_p(l\cdot s_0)=\gamma_0 g_0(l\cdot s_0)\mathrm{e}^{-\mathrm{j}l\theta_p}\,. \tag{4}$$

For $P$ sources with the complex weights $\gamma_p$ the virtual aperture function of the array output can be described with

$$g(l\cdot s_0)=\sum_{p=1}^{P}\gamma_p g_0(l\cdot s_0)\mathrm{e}^{-\mathrm{j}l\theta_p}\,. \tag{5}$$

That yields

$$g(l\cdot s_0)=g_0(l\cdot s_0)\sum_{p=1}^{P}\gamma_p\,\mathrm{e}^{-\mathrm{j}l\theta_p}\,. \tag{6}$$

With the array response matrix $\mathbf{A}=[\mathbf{a}(\theta_1),\cdots,\mathbf{a}(\theta_P)]$ that is composed of the complex exponential vectors related to the $P$ source directions $\mathbf{a}(\theta_p)=[1,\exp(-\mathrm{j}\theta_p),\cdots,\exp(-\mathrm{j}\theta_p(N-1))]^T$, the diagonal matrix $\mathbf{G}_0=\mathrm{diag}\{g_0(0),\cdots,g_0((N-1)s_0)\}$ that contains the element virtual aperture function, the complex weight vector $\boldsymbol{\gamma}=[\gamma_1,\cdots,\gamma_P]^T$, and the virtual aperture output vector $\mathbf{g}=[g(0),g(s_0),\cdots,g((N-1)s_0)]$ equation (6) can be rewritten as

$$\mathbf{g}=\mathbf{G}\mathbf{A}\boldsymbol{\gamma}\,. \tag{7}$$

Since the array output signal is given by samples of the beam pattern $b_0(n\theta_0)$, the Nyquist sampling criterion has to be considered. I.e., the virtual aperture function $g\ (l\cdot s_0)$ contains the required rotational invariance structure only if the Fourier transform of $b_0(\theta)$ has finite support in $Ls_0$ and if $N\geq L$. The virtual aperture output vector is calculated from the received data $\mathbf{b}=\mathbf{B}\boldsymbol{\gamma}$ by DFT where $\mathbf{B}=[\mathbf{b}(\theta_1),\cdots,\mathbf{b}(\theta_p)]$ is a matrix that contains the $P$ beam vectors $\mathbf{b}(\theta_p)=[b_p(0),\cdots,b_p((N-1)\cdot\theta_0)]^T$ .The transformation matrix $\mathbf{T}$ that transforms $\mathbf{b}$ into the modified aperture space

$$\mathbf{x}=\mathbf{T}\,\mathbf{b} \tag{8}$$

is defined as:

$$\mathbf{T}=\mathbf{G}_L^{-1}\mathbf{J}_L\mathbf{F}_N\,. \tag{9}$$

$\mathbf{F}_N$ is the $N$ x $N$ DFT matrix and $\mathbf{J}_L$ denotes a selection matrix

$$\mathbf{J}_L=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & 1 & 0\end{bmatrix}^{L\times N},$$

that chooses only the useful part of $g_0(l\cdot s_0)$ and ensures a good condition of $\mathbf{G}_L=\mathbf{J}_L\mathbf{G}_0$, and thus also of its inverse $\mathbf{G}_L^{-1}$.

With the measures described, the modified virtual aperture space $\mathbf{x}$ shows the required rotational invariance structure. Therefore, the signal subspace can be derived from $\mathbf{x}$ as it is well known from the ESPRIT algorithm and even the usual sub-array smoothing techniques can be applied. Let $\mathbf{x}_k=[x_k,x_{k+1},\ldots,x_{k+S-1}]^T$ denote a part of $\mathbf{x}$ beginning with the element $k$ and a length of S. Then the data matrix $\mathbf{X}_S^f=[\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_M],M=L-S+1$ is build using forward smoothing and with the reflection matrix $\boldsymbol{\Pi}$ the data matrix $\mathbf{X}_S^b=\boldsymbol{\Pi}\mathbf{X}_S^{f*}$ is build using backward smoothing. Usually both data matrices are combined to the full data matrix $\mathbf{X}_S=[\mathbf{X}_S^f\quad\mathbf{X}_S^b]$. The signal subspace $\mathbf{E}_s$ for ESPRIT may be calculated via the left singular vectors belonging to the $P$ largest singular values of

$$\mathbf{X}_S=[\mathbf{E}_S\quad\mathbf{E}_N]\begin{bmatrix}\Sigma_S & \mathbf{0}\\ \mathbf{0} & \Sigma_N\end{bmatrix}\begin{bmatrix}\mathbf{K}_S^H\\ \mathbf{K}_N^H\end{bmatrix}.$$

The signal subspace $\mathbf{E}_S$ is related to the DoA matrix $\mathbf{A}$ according to $\mathbf{E}_S=\mathbf{A}\,\mathbf{R}$, where $\mathbf{R}$ is under the stated conditions a unknown full rank matrix of size $P\times P$. With the ESPRIT [2] invariance equation $\mathbf{J}_1\mathbf{A}\boldsymbol{\Omega}=\mathbf{J}_2\mathbf{A}$ and substituting $\mathbf{A}=\mathbf{E}_S\,\mathbf{R}^{-1}$ leads to

$$\mathbf{J}_1\mathbf{E}_S\boldsymbol{\Psi}=\mathbf{J}_2\mathbf{E}_S\,, \tag{12}$$

with $\mathbf{\Psi} = \mathbf{R}^{-1}\mathbf{\Omega}\,\mathbf{R}$. Thus, the angles belonging to the eigenvalues of the $P$ x $P$ solution $\mathbf{\Psi}$ to the $(S-1)\times P$ matrix equations in (12) contain the directions to be estimated. Since **A** is conjugate centrosymmetric, the *unitary* ESPRIT [4] can preferably be used for a computationally effective implementation.

Table 1: *Summary of CUBA-ESPRIT*

1) Compute $\mathbf{x} = \mathbf{T}\,\mathbf{b}$.
2) Compute $\mathbf{X}_S$ from **x**.
3) Compute $\mathbf{E}_S$ via the left singular vectors belonging to the P largest singular values of $\mathbf{X}_s$.
4) Compute the solution to $(\mathbf{J}_1\mathbf{E}_S)\mathbf{\Psi} = (\mathbf{J}_2\mathbf{E}_S)$.
5) Compute $\theta_i$, $i$=1...$P$, as the angles of the complex eigenvalues of $\mathbf{\Psi}$, the $\theta_i$ are already the estimated directions.

Obviously, the ideal beam pattern of a CUBA for the described algorithm is a sinc-function, since a beam pattern $b_0(\theta) = \mathrm{sinc}(L/2\cdot\theta)$ provides a strictly virtual aperture limited function $g_0(l\cdot s_0) = \mathrm{rect}(l/L)$. In the context of Nyquist sampling, this is the ideal anti-aliasing lowpass filter for an angular sampling rate of $1/L\cdot s_0$. For a given $L$, the maximum number of resolvable coherent sources is $P_{\max} = 2/3\,L$, the number of resolvable incoherent sources is $P_{\max} = L-1$.

## A CUBA CALIBRATION ALGORITHM

Every real antenna array is characterized by non-uniform beam patterns, unknown phase factors, inter-element coupling, and so on. Hence, each antenna array should be calibrated to minimize DoA estimation errors. In [8] a calibration algorithm is described that results in an estimation of the array error matrix **C**. But for the CUBA ESPRIT algorithm, the explicit knowledge of the array error matrix is not really necessary. Rather, the transformation of the disturbed array output signals $\tilde{\mathbf{b}} = \mathbf{C}\,\mathbf{b}$ into $\mathbf{x}_L = \mathbf{A}_L\boldsymbol{\gamma}$ is required. If $g_0(l\cdot s_0)$ is aperture limited, the transformation matrix

$$\mathbf{T} = \left(\mathbf{G}_L^{-1}\mathbf{J}_L\mathbf{F}\,\mathbf{C}^{-1}\right) \tag{16}$$

satisfies

$$\mathbf{x} = \mathbf{T}\,\tilde{\mathbf{b}}\,. \tag{17}$$

In reality, **T** can only be estimated from measured data. Using the ideas in [8], the following least squares minimization problem can be established

$$\min_{\hat{\mathbf{T}}} \sum_{k=1}^{K}\sum_{i=1}^{L-1} \mathbf{w}_{ik}^H\hat{\mathbf{T}}\tilde{\mathbf{b}}_k\tilde{\mathbf{b}}_k^H\hat{\mathbf{T}}^H\mathbf{w}_{ik}\,. \tag{18}$$

The $\mathbf{b}_k$'s are the array response vectors of a single source with k different incident angles $\theta_k$. The vectors $\mathbf{w}_{ik}$ are the ($L$-1) known nullsteering vectors related to the desired vector $\mathbf{a}_L(\theta_k)$ and $\hat{\mathbf{T}}$ is the $L$ by $N$ transformation matrix to be estimated. Using the matrix-vector exchange of **w** and the non-square matrix $\hat{\mathbf{T}}$, (18) changes to

$$\hat{\mathbf{t}} = \arg\min_{\hat{\mathbf{t}}}\ \hat{\mathbf{t}}\left(\sum_{k=1}^{K}\sum_{i=1}^{L-1}\mathbf{W}_{ik}^H\tilde{\mathbf{b}}_k\tilde{\mathbf{b}}_k^H\mathbf{W}_{ik}\right)\hat{\mathbf{t}}^H\,. \tag{19}$$

Which in turn leads to the minimization problem

$$\hat{\mathbf{t}} = \arg\min_{\hat{\mathbf{t}}}\ \hat{\mathbf{t}}\,\mathbf{R}\,\hat{\mathbf{t}}^H \tag{20}$$

with $\mathbf{R} = \mathbf{X}\cdot\mathbf{X}^H$ and $\mathbf{X} = \left[\mathbf{W}_{11}^H\tilde{\mathbf{b}}_1,\cdots,\mathbf{W}_{(L-1)1}^H\tilde{\mathbf{b}}_1,\cdots,\mathbf{W}_{(L-1)K}^H\tilde{\mathbf{b}}_K\right]$.

A least squares solution to (20) can be typically calculated via the left singular vector related to the smallest non zero singular value of **R** or **X**. In respect to numerical stability, the use of **X** is preferable. The optimal transformation matrix can be constructed by reshaping the vector $\hat{\mathbf{t}}$ to the matrix $\hat{\mathbf{T}}$ according to the used matrix-vector exchange of **w** and **W**.

## SIMULATION RESULTS

The following simulation results are based on measured beam patterns of a 6 element CUBA . The left picture of figure 2 shows the uncalibrated and the calibrated beam pattern $b_0(\theta)$ of a single antenna. The right picture depicts the respective virtual aperture functions. The transformation matrix has been estimated using the proposed calibration algorithm. The calibrated beam pattern is calculated using the inverse Fourier transform.

To verify the DoA estimation performance of the CUBA ESPRIT, a simple scenario with 3 coherent sources impinging at the array with the same power has been simulated. With 2 sources at fixed angles (-90°,+90°), and simultaneously rotating one source and the array from –75° to +75°, 151 output vectors have been created. Figure (3) illustrates the estimated angles.

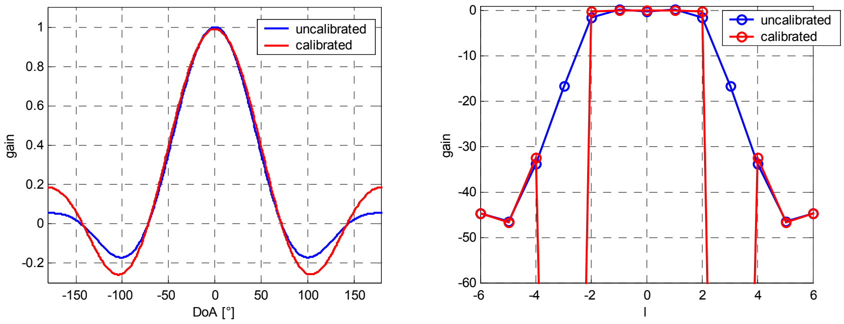


Fig. 2. CUBA beam pattern (left) and corresponding virtual aperture function (right)

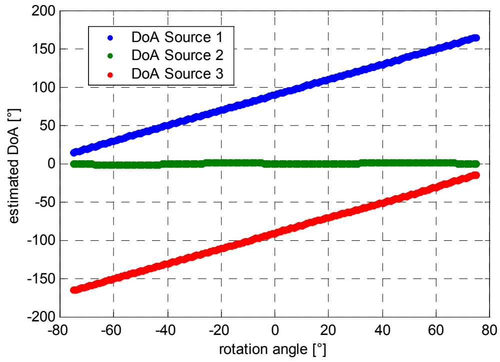


Fig. 3. CUBA ESPRIT angle estimation results for three coherent sources

## CONCLUSION

A CUBA-ESPRIT algorithm for circular uniform beam arrays has been developed. Since the algorithm can handle coherent sources, it can preferably be applied in mobile communication. The optimal beam pattern of a single array element for the developed DoA estimation algorithm is a sinc-function. The design of the beam pattern dictates the minimal number of array elements required and the number of resolvable sources.

The CUBA-ESPRIT and the proposed calibration algorithm have been successfully tested with a multi-beam antenna in combination with the RUSK-ATM Vector-Channel-Sounder.

## ACKNOWLEDGEMENT

This work has been supported by the "Deutsche Forschungsgemeinschaft". The authors wish to thank Prof. Wiesbeck and Dipl.-Ing. Zwick for providing their multi-beam antenna [6].

## REFERENCES

[1] A.J. van der Veen, "Algebraic Methods for Deterministic Blind Beamforming", Proc. IEEE, Special Issue on: Blind System Indentification and Estimation, Oct. 1998

[2] R. Roy, T. Kailath, „ESPRIT-Estimation of signal parameters via rotational invariance techniques", IEEE Trans. Acoust., Speech, Signal Processing, vol. 37, pp 984-995, July 1989.

[4] M. Haard, J.A. Nossek, „Unitary ESPRIT: how to obtain increased estimation accuracy with a reduced computational burden", IEEE Trans. Signal Processing, vol.43, pp. 1232-1242, May 1995.

[5] R.S. Thomä, D. Hampicke, A. Richter, G. Sommerkorn, A. Schneider, U. Trautwein, "Identification of Time-Variant Directional Radio Channels"; Proc. 16th IEEE Instrumentation and Measurement Technology Conference (IMTC'99); Venice/Italy, vol. 1, pp. 176-181, 1999.

[6] F. Demmerle, S. Kern, W. Wiesbeck, "A Bi-conical Multibeam Antenna for Space Division Multiple Access", IEEE AP-S, Montreal, July 13 - 18, 1997, pp 1082 – 1085.

[7] S.U.Pillai, "Array Signal Processing", Springer-Verlag New York Inc., 1989, pp. 47-59

[8] K. Pensel, J.A. Nossek: "Uplink and Downlink Calibration of an Antenna Array in a Mobile Communication System", COST 259, 3rd Meeting, 1997, September 24-26, TD-58, Lisbon, Portugal.